The next-to-leading order gravitational spin(1)-spin(2) dynamics in Hamiltonian form

Jan Steinhoff, Steven Hergt, and Gerhard Schäfer
Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität, Max-Wien-Pl. 1, 07743 Jena, Germany
(Dated: May 20, 2008)

Based on recent developments by the authors a next-to-leading order spin(1)-spin(2) Hamiltonian is derived for the first time. The result is obtained within the canonical formalism of Arnowitt, Deser, and Misner (ADM) utilizing their generalized isotropic coordinates. A comparison with other methods is given.

PACS numbers: 04.25.-g, 04.25.Nx

I. INTRODUCTION

In this paper we present the next-to-leading (NLO) order gravitational spin(1)-spin(2) dynamics in Hamiltonian form. The result is based on the ADM canonical formalism [1] for spinning classical objects recently derived by Steinhoff, Schäfer, and Hergt [2] which already has shown its power by the derivation of the Hamiltonian of two spinning compact bodies with next-to-leading order gravitational spin-orbit coupling, lately obtained by Damour, Jaranowski, and Schäfer [3].

The following notations will be used throughout the paper: The canonical spin tensor of the a-th object (particle) is the euclidean spin tensor $S_a^{ij} = S_{aij} = \epsilon^{ijk} S_a^k$, $\mathbf{S}_a = (S_a^k) = (S_{ak})$, and it also holds by definition $\hat{S}_{aij} = e_{ik} e_{jl} S_{akl}$ with e_{ik} being the symmetric root of the symmetric 3-metric γ_{ik}. $\hat{S}_{aij}$ and S_{aij} fulfill the conserved-length relations $\hat{S}_{aij}\gamma^{ik}\gamma^{jl}\hat{S}_{akl} = S_{aij}S_{aij} = \text{const.}$ The mass parameter of the a-th particle is denoted m_a. The 4-vector n^μ is the unit vector orthogonal to the spacelike hypersurfaces $t = \text{const.}$; its components are $n_\mu = (-N, 0, 0, 0)$, where N is the lapse function. The short-cut notation $-np_a$ is used for $\sqrt{m_a^2 + \gamma^{ij} p_{ai} p_{aj}}$, where $(p_{ai}) = \mathbf{p}_a$ denotes the canonical momentum of the a-th particle. The canonical particle position variables are $(x_a^i) = \mathbf{x}_a$ and the velocities read $\mathbf{v}_a = (v_a^i) = (\dot{x}_a^i)$, where the dot means coordinate time derivative.

Our units are $c = 1$, where c is the velocity of light. G will denote the Newtonian gravitational constant. Greek indices will run over $0, 1, 2, 3$, Latin over $1, 2$ if from the beginning of the alphabet and $1, 2, 3$ if from the middle. For the signature of spacetime we choose $+2$.

II. SPINNING OBJECTS IN THE ADM FORMALISM

Recently in [2] it has been shown that the matter source parts of the energy and momentum constraint equations, respectively $\mathcal{H}^{\rm M}$ and $\mathcal{H}_i^{\rm M}$, are given in terms of canonical position, momentum, and spin variables in the form, to the post-Newtonian orders indicated,

$$\mathcal{H}^{\rm M} = \sum_a \left[-np_a \delta_a \left\{ 1 + \frac{1}{2} \frac{p_{aj} \hat{S}_{ali} \gamma^{lk} \gamma^{ij}_{\ ,k}}{(np_a)^2} \right\} + \left(\frac{p_{al} \hat{S}_{aij} \gamma^{il} \gamma^{kj} \delta_a}{m_a - np_a} \right)_{,k} \right] + \mathcal{O}\left(S/c^6\right), \tag{2.1}$$

$$\mathcal{H}_i^{\rm M} = \sum_a \left[p_{ai} \delta_a + \frac{1}{2} \left(S_{aik} + \frac{p_{al} S_{al(i} p_{ak)}}{m_a^2} \right) \delta_{a,k} \right] + \mathcal{O}\left(S/c^4\right). \tag{2.2}$$

These expressions are sufficient for the derivation of the Hamiltonian in [3]. The applied equal-time Poisson brackets read

$$\{x_a^i, p_{aj}\} = \delta_{ij}, \qquad \{S_a^{ij}, S_a^{kl}\} = \delta_{ik} S_a^{jl} - \delta_{jk} S_a^{il} - \delta_{il} S_a^{jk} + \delta_{jl} S_a^{ik}, \qquad \text{zero otherwise}. \tag{2.3}$$

It is important to note that the expressions for $\mathcal{H}^{\rm M}$ and $\mathcal{H}_i^{\rm M}$ are given in the ADM transverse-traceless (ADMTT) gauge which refers to generalized isotropic coordinates defined by the conditions

$$\gamma_{ij} = \left(1 + \frac{1}{8}\phi \right)^4 \delta_{ij} + h_{ij}^{\rm TT}, \quad \pi^{ii} = 0, \tag{2.4}$$

with h^{TT}_{ij} being transverse and traceless ($h^{\mathrm{TT}}_{ij,j} = 0$, $h^{\mathrm{TT}}_{ii} = 0$). The square root of the metric takes the form, to sufficient approximation for the Hamiltonian,

$$e_{ij} = \left(1 + \frac{1}{4}\phi\right)\delta_{ij} + \mathcal{O}\left(\phi^2, h^{\mathrm{TT}}_{ij}\right) . \tag{2.5}$$

In this approximation, $\hat{S}_{ai}{}^{j} \equiv \hat{S}_{ail}\gamma^{lj} = \hat{S}_{alj}\gamma^{li} \equiv \hat{S}^{i}_{aj} = S_{aij} = S^{ij}_{a}$ holds.

The canonical conjugate to h^{TT}_{ij}, $\frac{1}{16\pi G}\pi^{ij}_{\mathrm{TT}}$, i.e.,

$$\frac{1}{16\pi G}\{h^{\mathrm{TT}}_{ij}(\mathbf{x},t), \pi^{kl}_{\mathrm{TT}}(\mathbf{x}',t)\} = \delta^{\mathrm{TT}kl}_{ij}(\mathbf{x}-\mathbf{x}')\,, \qquad \text{zero otherwise}\,, \tag{2.6}$$

will play no role in the calculations of the present paper. Only the longitudinal part of π^{ij}, $\tilde{\pi}^{ij}$, contributes (notice: $\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij}_{\mathrm{TT}}$). Thus, the ADM Hamiltonian H will not depend on π^{ij}_{TT}, i.e.,

$$H[\mathbf{x}_a, \mathbf{p}_a, h^{\mathrm{TT}}_{ij}] = -\frac{1}{16\pi G}\int \mathrm{d}^3x\, \Delta\phi\,. \tag{2.7}$$

Therefore, h^{TT}_{ij} is allowed to be replaced by matter variables (after solution of the evolution equations which in our approximation is an elliptic equation only) to get an autonomous Hamiltonian. Otherwise the transition to a Routhian would have to be performed, see [4]. The equations to be solved for the obtention of the autonomous matter Hamiltonian read, on the one side

$$\frac{1}{\sqrt{\gamma}}\left(-\gamma_{ik}\gamma_{jl}\pi^{ij}\pi^{kl} + \frac{1}{2}\gamma_{ij}\gamma_{kl}\pi^{ij}\pi^{kl}\right) + \sqrt{\gamma}R_{(3)} = 16\pi G\mathcal{H}^{\mathrm{M}}\,, \tag{2.8}$$

$$-\gamma_{ij}\pi^{jk}{}_{;k} = 8\pi G\mathcal{H}^{\mathrm{M}}_{i}\,, \tag{2.9}$$

where $R_{(3)}$ denotes the Ricci scalar of the $t = $ const. slices, γ is the determinant of the 3-metric, and ; the 3-dim. covariant derivative, and on the other side

$$0 = \{H, \pi^{ij}_{\mathrm{TT}}\} + \mathcal{O}\left(1/c^6\right). \tag{2.10}$$

After tedious calculations, the NLO spin(1)-spin(2) interaction part of the Hamiltonian results in

$$\begin{aligned}
H^{\mathrm{NLO}}_{\mathrm{SS}} = {} & \frac{1}{2m_1m_2r^3_{12}}[\tfrac{3}{2}((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_2\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) + 6((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) \\
& - 15(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) - 3(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{p}_2) \\
& + 3(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) + 3(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) + 3(\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) \\
& + 3(\mathbf{S}_2\cdot\mathbf{p}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) - \tfrac{1}{2}(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{p}_1) + (\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{p}_2) \\
& - 3(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) + \tfrac{1}{2}(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)] \\
& + \frac{3}{2m^2_1r^3_{12}}[-((\mathbf{p}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12})((\mathbf{p}_1\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) + (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{p}_1\cdot\mathbf{n}_{12})] \\
& + \frac{3}{2m^2_2r^3_{12}}[-((\mathbf{p}_2\times\mathbf{S}_2)\cdot\mathbf{n}_{12})((\mathbf{p}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12}) + (\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})] \\
& + \frac{6(m_1+m_2)}{r^4_{12}}[(\mathbf{S}_1\cdot\mathbf{S}_2) - 2(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})]\,,
\end{aligned} \tag{2.11}$$

where $r_{12} = |\mathbf{x}_1 - \mathbf{x}_2|$ is the euclidean distance between the two particles and $\mathbf{n}_{12}$ denotes the unit vector $r_{12}\mathbf{n}_{12} = \mathbf{x}_1 - \mathbf{x}_2$.

III. DIFFERENT DERIVATION OF THE SPIN(1)-SPIN(2) HAMILTONIAN

We follow here the procedure described in [3]. The implementation of spin into the Eq. (4.9) of [3] results in

$$v^{i\,\mathrm{spin}}_{(3)a} = -\sum_{b\neq a}\left(\frac{3m_bS_{aij}}{2m_a} + 2S_{bij}\right)\frac{n^j_{ab}}{r^2_{ab}}\,. \tag{3.1}$$

The NLO spin(1)-spin(2) interation Hamiltonian is given by

$$H_{\rm SS}^{\rm NLO} = \tilde{\Omega}_{(4)ij}\, S_1^i\, S_2^j = \mathbf{\Omega}_{(4)}^{\rm spin(2)} \cdot \mathbf{S}_1 = \mathbf{\Omega}_{(4)}^{\rm spin(1)} \cdot \mathbf{S}_2\,. \tag{3.2}$$

Using Eq. (4.10b) in Ref. [3], but calculated for metric functions resulting from our matter source terms, the obtained Hamiltonian coincides with the one calculated in the present paper.

In this completely independent approach also the matter source part $\frac{1}{2}N\gamma^{ik}\gamma^{jl}T_{kl}$ of the evolution equations contributes,

$$T_{kl} = \sum_a \left[-\frac{p_{ak}p_{al}}{np_a}\delta_a - \frac{S_{aj(k}p_{al)}}{m_a}\delta_{a,j} \right] + \mathcal{O}\left(S/c^2\right), \tag{3.3}$$

and lapse and shift functions have to be determined too. Details of the calculations can be found in [2].

IV. CONSISTENCY

The correctness of the derived spin(1)-spin(2) Hamiltonian can best be verified by the construction of the global Poincaré algebra. The generators of the global Poincaré algebra are the total linear momentum $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$, the total angular momentum $\mathbf{J} = \mathbf{x}_1 \times \mathbf{p}_1 + \mathbf{x}_2 \times \mathbf{p}_2 + \mathbf{S}_1 + \mathbf{S}_2$, the total Hamiltonian H, and the total center-of-mass generator $\mathbf{G}$. The latter object is defined by $\mathbf{G} = -\frac{1}{16\pi G}\int \mathrm{d}^3x\, \mathbf{x}\Delta\phi$, e.g., see [5], and turns out to be

$$\mathbf{G} = \mathbf{G}_{\rm PM} + \mathbf{G}_{\rm SO} + \frac{G}{2r_{12}^2}\left[(\mathbf{S}_2\cdot\mathbf{n}_{12})\mathbf{S}_1 - (\mathbf{S}_1\cdot\mathbf{n}_{12})\mathbf{S}_2 + (3(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{n}_{12}) - (\mathbf{S}_1\cdot\mathbf{S}_2))\,\frac{\mathbf{x}_1+\mathbf{x}_2}{r_{12}} \right], \tag{4.1}$$

where $\mathbf{G}_{\rm SO}$ denotes the NLO spin-orbit coupling contribution as given in [3] and $\mathbf{G}_{\rm PM}$ is the point-mass part from [6]. It is straightforward to show that the above generators do fulfill the Poincaré algebra.

V. COMPARISON WITH OTHER METHODS AND RESULTS

In a recent paper [7], based on Ref. [8], Porto and Rothstein derived a next-to-leading order spin(1)-spin(2) potential using an action approach with spin supplementary conditions imposed on the action level. A consistency calculation, however, which would have shown that the spin supplementary conditions are preserved under the variational principle, and thus under the equations of motion, has not been undertaken nor has it been shown that, in contrast to their claim, the used position, velocity, and spin variables are those that relate to canonical ones in standard manner, cf., e.g., [9]. A consistency check of their intuitive canonical approach by an independent method is therefore necessary, which is the subject of this Section.

The relevant part of the Lagrangian of Porto and Rothstein reads $L^{\rm PR} = \frac{1}{2}m_1\mathbf{v}_1^2 + \frac{1}{2}m_2\mathbf{v}_2^2 - V_N - V_{\rm SO}^{\rm LO} - V_{\rm SS}^{\rm PR}$ where V_N is the Newtonian potential, $V_{\rm SS}^{\rm PR}$ is given by Eq. (12) in [7] and the leading order spin-orbit coupling potential function $V_{\rm SO}^{\rm LO}$ results from, e.g., [3],

$$V_{\rm SO}^{\rm LO} = \frac{G}{r_{12}^2}\left[\frac{3}{2}m_2((\mathbf{v}_1\times\mathbf{S}_1)\cdot\mathbf{n}_{12}) - 2m_2((\mathbf{v}_2\times\mathbf{S}_1)\cdot\mathbf{n}_{12}) - \frac{3}{2}m_1((\mathbf{v}_2\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) + 2m_1((\mathbf{v}_1\times\mathbf{S}_2)\cdot\mathbf{n}_{12}) \right]. \tag{5.1}$$

The canonical momenta are given by $\mathbf{p}_a = \frac{\partial L^{\rm PR}}{\partial \mathbf{v}_a}$, i.e.,

$$\mathbf{p}_1 = m_1\mathbf{v}_1 - \frac{G}{r_{12}^2}\left[\frac{3}{2}m_2(\mathbf{S}_1\times\mathbf{n}_{12}) + 2m_1(\mathbf{S}_2\times\mathbf{n}_{12}) \right] - \frac{\partial V_{\rm SS}^{\rm PR}}{\partial \mathbf{v}_1}, \tag{5.2}$$

$$\mathbf{p}_2 = m_2\mathbf{v}_2 + \frac{G}{r_{12}^2}\left[\frac{3}{2}m_1(\mathbf{S}_2\times\mathbf{n}_{12}) + 2m_2(\mathbf{S}_1\times\mathbf{n}_{12}) \right] - \frac{\partial V_{\rm SS}^{\rm PR}}{\partial \mathbf{v}_2}\,. \tag{5.3}$$

The Hamiltonian of Porto and Rothstein then takes the form $H^{\rm PR} = H_N + H_{\rm SO}^{\rm LO} + H_{\rm SS}^{\rm PR}$. We can get $H^{\rm PR}$ by replacing the velocities by canonical momenta in the following expression

$$H^{\rm PR} = \mathbf{v}_1\cdot\mathbf{p}_1 + \mathbf{v}_2\cdot\mathbf{p}_2 - L^{\rm PR}\,. \tag{5.4}$$

Note that the $\frac{\partial V_{\rm SS}^{\rm PR}}{\partial \mathbf{v}_a}$ terms do not contribute. The difference between $H_{\rm SS}^{\rm NLO}$ and $H_{\rm SS}^{\rm PR}$ reads

$$\begin{aligned}\delta H_{\rm SS}^{\rm NLO} = \frac{G}{2m_1 m_2 r_{12}^3}&[3(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) + 3(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})\\ &- 2(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{p}_1) - 6(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12}) + 2(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{p}_2)]\\ &+ \frac{G}{2m_1^2 r_{12}^3}[3(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_1\cdot\mathbf{S}_2)\mathbf{p}_1^2 - 3(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{p}_1\cdot\mathbf{n}_{12}) + (\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{p}_1)]\\ &+ \frac{G}{2m_2^2 r_{12}^3}[3(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})^2 - (\mathbf{S}_1\cdot\mathbf{S}_2)\mathbf{p}_2^2 - 3(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12}) + (\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{p}_2)]\,.\end{aligned} \tag{5.5}$$

Obviously, there is agreement at $\mathcal{O}(G^2)$.

There should exist an infinitesimal generator g for a canonical transformation such that $\delta H_{\rm SS}^{\rm NLO} = \{H_N, g\}$. Plugging in the ansatz

$$\begin{aligned} g = a\frac{G}{r_{12}^2}&\left[\frac{1}{m_1}(\mathbf{S}_1\cdot\mathbf{p}_1)(\mathbf{S}_2\cdot\mathbf{n}_{12}) - \frac{1}{m_2}(\mathbf{S}_2\cdot\mathbf{p}_2)(\mathbf{S}_1\cdot\mathbf{n}_{12})\right]\\ &+ b\frac{G}{r_{12}^2}\left[\frac{1}{m_2}(\mathbf{S}_1\cdot\mathbf{p}_2)(\mathbf{S}_2\cdot\mathbf{n}_{12}) - \frac{1}{m_1}(\mathbf{S}_2\cdot\mathbf{p}_1)(\mathbf{S}_1\cdot\mathbf{n}_{12})\right]\\ &+ c\frac{G}{r_{12}^2}\left[\frac{1}{m_1}(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_1\cdot\mathbf{n}_{12}) - \frac{1}{m_2}(\mathbf{S}_1\cdot\mathbf{S}_2)(\mathbf{p}_2\cdot\mathbf{n}_{12})\right]\\ &+ d\frac{G}{r_{12}^2}\left[\frac{1}{m_1}(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_1\cdot\mathbf{n}_{12}) - \frac{1}{m_2}(\mathbf{S}_1\cdot\mathbf{n}_{12})(\mathbf{S}_2\cdot\mathbf{n}_{12})(\mathbf{p}_2\cdot\mathbf{n}_{12})\right]\end{aligned} \tag{5.6}$$

and comparing $\mathcal{O}(G)$ terms gives

$$a = 0\,, \quad b = \frac{1}{2}\,, \quad c = \frac{1}{2}\,, \quad d = 0\,. \tag{5.7}$$

The vanishing of $\mathcal{O}(G^2)$ terms yields

$$c = 0\,, \quad -a + b - d = 0\,, \tag{5.8}$$

which is incompatible with the canonical transformation that is needed for the $\mathcal{O}(G)$ terms.

In Ref. [10], which is a short reply to the first version of the present paper [11], Porto and Rothstein pointed out that their result in [7] is incomplete in the sense that it only includes contributions from spin-spin *diagrams*. It was not realized in [7] that spin-orbit diagrams also contribute to the next-to-leading order spin(1)-spin(2) interaction. If these contributions are included, the canonical transformation defined by (5.6) and (5.7) leads to an additional agreement at $\mathcal{O}(G^2)$, and thus to full agreement. It should be recalled that our derivation is completely different and includes all contributions to the spin(1)-spin(2) interaction from the very beginning.

Acknowledgments

This work is supported by the Deutsche Forschungsgemeinschaft (DFG) through SFB/TR7 "Gravitational Wave Astronomy".

[1] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), p. 227, arXiv:gr-qc/0405109.

[2] J. Steinhoff, G. Schäfer, and S. Hergt, *The ADM canonical formalism for gravitating spinning objects*, Phys. Rev. D, submitted.

[3] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **77**, 064032 (2008).

[4] P. Jaranowski and G. Schäfer, Phys. Rev. D **55**, 4712 (1997); **57**, 7274 (1998).

[5] T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) **88**, 286 (1974).

[6] T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **62**, 021501(R) (2000); **63**, 029903(E) (2001).

[7] R. A. Porto and I. Z. Rothstein, Phys. Rev. Lett. **97**, 021101 (2006), arXiv:gr-qc/0604099v1.

[8] R. A. Porto, Phys. Rev. D **73**, 104031 (2006).

[9] K. Yee and M. Bander, Phys. Rev. D **48**, 2797 (1993).

[10] R. A. Porto and I. Z. Rothstein, *Comment on 'On the next-to-leading order gravitational spin(1)-spin(2) dynamics' by J. Steinhoff et al.*, arXiv:0712.2032v1 [gr-qc].

[11] J. Steinhoff, S. Hergt, and G. Schäfer, *On the next-to-leading order gravitational spin(1)-spin(2) dynamics*, arXiv:0712.1716v1 [gr-qc].